Generalized Inverses of Substochastic Matrices

Cecil E. Robinson, Jr.
Department of Mathematics
University of Southern Mississippi
Hattiesburg, Mississippi 39406-5045

Submitted by David H. Carlson

ABSTRACT

This paper looks at the question of when a substochastic matrix has a substochastic generalized inverse. This question is answered for several generalized inverses, including semiinverses, the Moore-Penrose inverse, and the group inverse. Methods for constructing all such inverses are given.

1. INTRODUCTION

A real nonnegative matrix A is substochastic [stochastic] if the sum of the entries in each row is between zero and one inclusive [equal to one]. If both A and A^T, the transpose of A, are substochastic [stochastic], then A is doubly substochastic [stochastic]. Given any complex matrix A, any matrix X such that $AXA = A$ and $XAX = X$ is called a semiinverse of A. The unique semiinverse X of A such that AX and XA are Hermitian is called the Moore-Penrose inverse of A and is denoted by A^+. Properties of A^+ can be found in [1], [6], and [8] as well as in other such works. If A has a semiinverse X such that $AX = XA$, then X is called the group inverse and is denoted by $A^{\#}$. Note that when $A^{\#}$ exists, it is unique. Other properties of $A^{\#}$ can be found in [2, 3, 7, 13], as well as in other such works.

Plemmons and Cline [9] have shown that if a doubly stochastic matrix A has a doubly stochastic semiinverse, then A^T is the unique such semiinverse; moreover, $A^T = A^+$ in this case. The structure of S_n, the semigroup of all $n \times n$ stochastic matrices, has been studied by Schwarz in [11], where he characterized the maximal subgroups of S_n. Schwarz [12] and Farahat [5] independently discovered the situation for doubly stochastic matrices. These

52 Vanderbilt Ave., New York, NY 10017 0024-3795/87/$3.50

results were generalized by Robinson [10] for the substochastic case. All of these results concerning maximal subgroups can be obtained as corollaries to the theorems in this paper. Several of the questions answered in this paper have been answered for the stochastic case by Wall [14].

2. SEMIINVERSES

In this section we will consider the problem of when a substochastic matrix has a substochastic semiinverse. We will also describe a method of constructing all such semiinverses. First we have the following lemma, which describes all idempotent substochastic matrices.

LEMMA 1 (Robinson [10]). *An $n \times n$ substochastic matrix E of rank k is idempotent if and only if there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} E_1 \oplus \cdots \oplus E_k & 0 \\ [R_1, \dots, R_k] & 0 \\ [S_1, \dots, S_k] & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where each E_i is a positive $n_i \times n_i$ stochastic idempotent matrix of rank one; each R_i is an $n_{k+1} \times n_i$ matrix of the form $R_i = N_i E_i'$, with each N_i an $n_{k+1} \times n_{k+1}$ nonnegative diagonal matrix such that $N_1 + \cdots + N_k = I$, and each E_i' consisting of n_{k+1} rows each equal to a row of E_i; $n_1 + \cdots + n_{k+1} = h$, where h equals the number of stochastic rows of E, $n_1 \geqslant \cdots \geqslant n_k \geqslant 1$, and $n_{k+1} \geqslant 0$; and each S_i is an $m \times n_i$ matrix of the form $S_i = N_i' E_i''$, with each N_i' an $m \times m$ nonnegative diagonal matrix such that the diagonal entries of $N_1' + \cdots + N_k'$ are between zero and one noninclusive, and each E_i'' consists of m rows each equal to a row of E_i, where m equals the number of nonzero nonstochastic rows of E.

It should be noted here that not all of the blocks in the above matrix need appear in a given idempotent substochastic matrix. Also note that any rank one stochastic matrix has constant columns. The following lemma describes all idempotent doubly substochastic matrices.

LEMMA 2 (Robinson [10]). *An $n \times n$ doubly substochastic matrix E of rank k is idempotent if and only if there exists a permutation matrix P such* *that* $PEP^T = E_1 \oplus \cdots \oplus E_k \oplus 0$ *where*

(a) *each* E_i *is the* $n_i \times n_i$ *matrix having each entry equal to* $1/n_i$; *and*

(b) $n_1 + \cdots + n_k = h$, *where* h *equals the number of nonzero rows of* E, *and* $n_1 \geqslant \cdots \geqslant n_k \geqslant 1$.

The following theorem characterizes those substochastic matrices that have a substochastic semiinverse. Furthermore, it describes a method of constructing all such semiinverses.

THEOREM 3. *Let* A *be any* $n \times s$ *substochastic matrix of rank* k. *Then* A *has a substochastic semiinverse* X *if and only if there exist permutation matrices* P *and* Q *such that*

$$PAQ^T = \begin{bmatrix} B & 0 \\ C & 0 \\ D & 0 \\ 0 & 0 \end{bmatrix},$$

where

(a) B *has the partitioned form* $B = [B_{ij}]$, $i, j = 1, \dots, k$, *where each* B_{ij} *is* $n_i \times s_j$;

(b) *for each* i *and each* j, *exactly one block* B_{ij} *is nonzero and this block is a positive stochastic matrix of rank one;*

(c) $C = [C_1, \dots, C_k]$, *where each* $C_i = G_i B'_{ji}$ *is* $n_{k+1} \times s_i$, G_i *is a nonnegative* $n_{k+1} \times n_{k+1}$ *diagonal matrix,* $G_1 + \cdots + G_k = I$, *and* B'_{ji} *consists of* n_{k+1} *rows each equal to a row of the positive block* B_{ji}; *and*

(d) $D = [D_1, \dots, D_k]$, *where each* $D_i = H_i B''_{ji}$ *is* $n_{k+2} \times s_i$, n_{k+2} *equals the number of nonstochastic nonzero rows of* A, H_i *is a nonnegative* $n_{k+2} \times n_{k+2}$ *diagonal matrix, the diagonal entries of* $H_1 + \cdots + H_k$ *are between zero and one noninclusive, and* B''_{ji} *consists of* n_{k+2} *rows, each a row of the positive block* B_{ji}.

In this case, any $s \times n$ *substochastic matrix such that* QXP^T *has the form*

$$QXP^T = \begin{bmatrix} Y & 0 \\ W & 0 \\ Z & 0 \\ 0 & 0 \end{bmatrix},$$

with all of the following conditions holding, is a semiinverse of A:

(e) *Y has the partitioned from $Y=[Y_{ij}]$, $i,j=1,\dots,k$, where each Y_{ij} is $s_i \times n_j$.*

(f) *$Y_{ij} \neq 0$ if and only if $B_{ji} \neq 0$.*

(g) *If $Y_{ij} \neq 0$, then Y_{ij} is any positive stochastic matrix of rank one.*

(h) *$W=[W_1,\dots,W_k]$, where each $W_i = J_i Y'_{ji}$ is $s_{k+1} \times n_i$, J_i is any $s_{k+1} \times s_{k+1}$ nonnegative diagonal matrix such that $J_1 + \cdots + J_k = I$, and Y'_{ji} consists of S_{k+1} rows, each a row of the positive block Y_{ji}.*

(i) *$Z=[Z_1,\dots,Z_k]$, where each $Z_i = K_i Y''_{ji}$ is $s_{k+2} \times n_i$, each K_i is any $s_{k+2} \times s_{k+2}$ nonnegative diagonal matrix such that the diagonal entries of $K_1 + \cdots + K_k$ are between zero and one noninclusive, and Y''_{ji} consists of s_{k+2} rows, each of which is a row of the positive block Y_{ji}.*

Proof. Assume that A has a substochastic semiinverse X. Let $E = AX$ and $F = XA$. Note that E and F are $n \times n$ and $s \times s$ idempotent substochastic matrices, respectively. Also, both E and F have rank k. By Lemma 1 there exists an $n \times n$ permutation matrix P such that

$$PEP^T = \begin{bmatrix} E_1 \oplus \cdots \oplus E_k & 0 \\ [R_1,\dots,R_k] & 0 \\ [S_1,\dots,S_k] & 0 \\ 0 & 0 \end{bmatrix},$$

where E_i is a positive $n_i \times n_i$ stochastic matrix of rank one; $R_i = N_i E'_i$ is $n_{k+1} \times n_i$; and $S_i = N'_i E''_i$ is $n_{k+2} \times n_i$, where $n_{k+2} = m$, and N_i, N'_i, E'_i, E''_i, $n_1,\dots,n_k$, n_{k+1}, and m are as given in Lemma 1. Also, by Lemma 1, there exists an $s \times s$ permutation matrix Q such that

$$QFQ^T = \begin{bmatrix} F_1 \oplus \cdots \oplus F_k & 0 \\ [U_1,\dots,U_k] & 0 \\ [V_1,\dots,V_k] & 0 \\ 0 & 0 \end{bmatrix},$$

where F_i is a positive $s_i \times s_i$ stochastic matrix of rank one, $U_i = M_i F'_i$ is $s_{k+1} \times s_i$, $V_i = M'_i F''_i$ is $s_{k+2} \times s_i$, and M_i, M'_i, F'_i, F''_i, $s_1,\dots,s_{k+1}$, and $s_{k+2} = m'$ are as given in Lemma 1.